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Exact Estimates for Monotone Interpolation

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Let $0 = x_0 < x_1 < \cdots < x_n = 1$, $0 = y_0 < y_1 < \cdots < y_n \leqslant 1$ and let A, B, C be defined by (3); suppose further that $C \geqslant c_0 n^{-1}$. We prove that for some constant $c = c(c_0)$, depending only on c_0, there exists an algebraic polynomial P of degree $\leqslant cn \ln(e + A/B)$ for which $P(x_i) = y_i$, $i = 1,..., n$, and $P'(x) \geqslant 0$, $0 \leqslant x \leqslant 1$.

1. INTRODUCTION

Let $0 = x_0 < \cdots < x_n = 1$ and let $y_0, y_1,..., y_n$ be real numbers satisfying $y_{i-1} \neq y_i$, $i = 1, 2,..., n$. Under these assumptions Wolibner [1], Kammerer [2] and Young [3] proved the existence of an algebraic polynomial P such that

(1) $P(x_i) = y_i$, $i = 0, 1,..., n$,

(2) $P(x)$ is monotone decreasing on $[x_{i-1}, x_i]$ if $y_{i-1} > y_i$ and monotone increasing on $[x_{i-1}, x_i]$ if $y_{i-1} < y_i$, $i = 1,..., n$.

P is called a partially monotone interpolation polynomial. When $y_{i-1} < y_i$ for $i = 1,..., n$, it is called a monotone interpolation polynomial.

The papers [1–3] do not contain estimations on the degree of the polynomial P. Later on, in [4–6], such estimations were given in terms of n and of

$$A = \max_{1 \leqslant i \leqslant n} \Delta y_i = \max_{1 \leqslant i \leqslant n} (y_i - y_{i-1}), \qquad B = \min_{1 \leqslant i \leqslant n} \Delta y_i, \qquad C = \min_{1 \leqslant i \leqslant n} \Delta x_i \tag{3}$$

In [4] a monotone approximation polynomial is constructed of degree not exceeding $c_1 nA/B$, where c_1 is an absolute constant. This estimate is exact when A/B is bounded for every n by an absolute constant. However, if this is not so, this estimate leads to a rather high degree of the monotone interpolation polynomial.

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In [5] the result of [4] is generalized and in [6] an estimate for the degree of P was given which is exact in the particular case when $A/B \asymp n^\alpha$, $\alpha \geqslant 1$. In this case it is proved that there exists a monotone interpolation polynomial of a degree not exceeding $c_2 \alpha n \ln n$, where c_2 is an absolute constant.

In [7] an estimate similar to the one in [6] was found for the case $y_0 > y_1 > \cdots > y_p > y_{p+1} < \cdots < y_n$ and the interpolation is done by a polynomial P with $P'(x) \leqslant 0$ for $x \in [x_0, x_p]$, $P'(x) \geqslant 0$ for $x \in [x_p, x_n]$.

Together with the results about monotone and partially monotone interpolation, other results have appeared on monotone and partially monotone approximation.

Initially the problem of uniform approximation of monotone functions by monotone polynomials was considered. A basic result, achieved later in [8], is that the order by which a monotone function can be approximated by a monotone polynomial from H_n is $O(\omega(f; n^{-1}))$. Here Hn is the set of all algebraic polynomials of degree $\leqslant n$, and $\omega(f; \delta)$ is the modulus of continuity of f: $\omega(f; \delta) = \sup_{|x'-x''| \leqslant \delta} |f(x') - f(x'')|$. This result is remarkable since, although an additional condition is imposed on the approximating polynomial, it preserves the exact order of approximation of the well-known Jackson theorem.

The problem of partially monotone uniform approximation can be formulated as follows: A function f is given which is monotone in every one of a finite number of subintervals, changing from increasing to decreasing or vice versa exactly at the ends of the subintervals. What is the order of approximation of f by polynomials from H_n, the set of those algebraic polynomials of degree $\leqslant n$ with the same monotonicity pattern as f?

This problem has been studied in [9–13]. In [13] it is shown that the exact order of partial monotone approximation is again $O(\omega(f; n^{-1}))$.

In the present paper we shall relate the above-mentioned problems on monotone interpolation and approximation to the following theorem giving an exact answer to the question on the degree of monotone interpolation polynomials for arbitrary knots and monotone interpolation data.

THEOREM 1. *Let the points x_i, y_i satisfy $0 = x_0 < x_1 < \cdots < x_n \leqslant 1$, $0 = y_0 < y_1 < \cdots < y_n \leqslant 1$, let A, B, C be given by* (3) *and let $C \geqslant c_0 n^{-1}$, where c_0 is a constant. Then there exists a polynomial Q of degree $\leqslant cn \ln(e + A/B)$, which satisfies*

$$Q(x_i) = y_i, \qquad i = 1,\ldots, n; \; Q'(x) \geqslant 0 \qquad \textit{for} \quad x \in [0, 1].$$

Here $c = c(c_0)$ is a constant depending only on c_0.

We also have

THEOREM 2. *Let the points x_i, y_i satisfy $0 = x_0 < \cdots < x_k = 1/2 <$*

$x_{k+1} < x_n \leqslant 1$, $1 \geqslant y_0 > \cdots y_k = 0$, $0 < y_{k+1} < \cdots < y_n \leqslant 1$. *If* $A' = \max_{1 \leqslant i \leqslant n} | \Delta y_i |$, $B' = \min_{1 \leqslant i \leqslant n} | \Delta y_i |$ *and* $C' = \min_{1 \leqslant i \leqslant n} \Delta x_i \geqslant c_0' n^{-1}$, *then there exists an algebraic polynomial* T *of degree* $\leqslant c' n \ln(e + A'/B')$ (c' *depending only on* c_0') *for which* $T(x_i) = y_i$, $0 \leqslant i \leqslant n$; $T'(x) \leqslant 0$ *for* $x \in [0, \frac{1}{2}]$ *and* $T'(x) \geqslant 0$ *for* $x \in [\frac{1}{2}, 1]$.

2. The Main Theorem

Lemma 1. *If the constants* b_i, ϵ_{ij} *satisfy* $b_i > 0$, $i = 1,..., n$, *and*

$$\max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | \epsilon_{ij} | (A + B)/B \leqslant M,$$

where $A = \max_{1 \leqslant i \leqslant n} b_i$, $B = \min_{1 \leqslant i \leqslant n} b_i$, *then the linear system*

$$\begin{bmatrix} M + \epsilon_{11} & \epsilon_{12} & \cdots & \epsilon_{1n} \\ \epsilon_{21} & M + \epsilon_{22} & \cdots & \epsilon_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \epsilon_{n1} & \epsilon_{n2} & \cdots & M + \epsilon_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}$$

has a unique positive solution (see [6] *where this theorem is given in a slightly different form*).

Proof. Since the system has a dominating main diagonal, it has a unique solution. Let $| x_l | = \max_{1 \leqslant i \leqslant n} | x_i |$. Then the lth equation assumes the form $M x_l + \sum_{j=1}^{n} \epsilon_{lj} x_j = b_l$. So

$$M | x_l | = \left| b_l - \sum_{j=1}^{n} \epsilon_{lj} x_j \right| \leqslant | b_l | + | x_l | \sum_{j=1}^{n} | \epsilon_{lj} |,$$

and therefore

$$| x_l | \leqslant A \Big/ \left(M - \sum_{j=1}^{n} | \epsilon_{lj} | \right) \leqslant A \Big/ \left(M - \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | \epsilon_{ij} | \right).$$

Now we show that the solution of the system is positive. Denote $x_k = \min_{1 \leqslant i \leqslant n} x_i$ and consider the kth equation. We obtain

$$\begin{aligned} M x_k &= b_k - \sum_{j=1}^{n} \epsilon_{kj} x_j \geqslant b_k - | x_l | \sum_{j=1}^{n} | \epsilon_{kj} | \\ &\geqslant B - \left[A \Big/ \left(M - \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | \epsilon_{ij} | \right) \right] \left(\max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} | \epsilon_{ij} | \right) > 0. \end{aligned}$$

The last inequality follows from the assumption.

Now let us carry out some preliminary calculations aimed at establishing Lemma 3 below, basic for the proof of Theorem 1.

Let m and r be natural numbers,

$$\begin{aligned} \sigma^T(x) &= 0, \qquad x \in [-\pi, -\pi/2), \\ &= 1, \qquad x \in [-\pi/2, \pi/2], \\ &= 0, \qquad x \in (\pi/2, \pi]; \end{aligned}$$

and

$$U_{m,r}(\sigma^T; x) = \mu \int_{-\pi}^{\pi} \sigma^T(x+t) \left(\frac{\sin(mt/2)}{m \sin(t/2)}\right)^{2r} dt,$$

where μ is defined by the condition

$$\mu \int_{-\pi}^{\pi} \left(\frac{\sin(mt/2)}{m \sin(t/2)}\right)^{2r} dt = 1. \tag{4}$$

The operator $U_{m,r}$ (see, e.g., [14, 15]) is called a generalized Jackson operator. It is known [15] that $U_{m,r}(\sigma^T; x)$ is a trigonometric polynomial of order $(m-1)r$.

From (4) it follows easily (see [15]) that

$$\mu \leqslant \frac{m}{2\pi} \left(\frac{\pi}{2}\right)^{2r}. \tag{5}$$

DEFINITION. A 2π-periodic function $f \in C[-\pi, \pi]$ is called bell-shaped if it is even and nondecreasing for $x \in [-\pi, 0]$.

In [16] it is proved that a generalized Jackson operator applied to bell-shaped step functions with jumps at the points $k\pi/m$, $k = 1,..., m-1$ yields, again, a bell-shaped function. The technique of this proof is similar to that in [8].

Since $\sigma^T(x)$ is a bell-shaped step function with jump at the point $\pi/2$ if m is even, $U_{m,r}$ is, in that case, also bell-shaped. In what follows we shall suppose m is even.

Since

$$U_{m,r}(\sigma^T; x) = \mu \int_0^{\pi} \left(\frac{\sin(mt/2)}{m \sin(t/2)}\right)^{2r} (\sigma^T(x+t) - \sigma^T(x-t))\, dt, \tag{6}$$

(4) yields

$$\begin{aligned} &U_{m,r}(\sigma^T; x) - \sigma^T(x) \\ &\quad = \mu \int_0^{\pi} \left(\frac{\sin(mt/2)}{m \sin(t/2)}\right)^{2r} (\sigma^T(x+t) - 2\sigma^T(x) + \sigma^T(x-t))\, dt. \end{aligned}$$

Let $0 \leqslant \alpha < \beta \leqslant \pi$. Then

$$\int_\alpha^\beta | U_{m,r}(\sigma^T; x) - \sigma^T(x)| \, dx$$

$$= \int_\alpha^\beta \left| \left[\mu \int_0^\delta \left(\frac{\sin(mt/2)}{m \sin(t/2)} \right)^{2r} (\sigma^T(x+t) - 2\sigma^T(x) + \sigma^T(x-t)) \, dt \right. \right.$$

$$\left. \left. + \mu \int_\delta^\pi \left(\frac{\sin(mt/2)}{m \sin(t/2)} \right)^{2r} (\sigma^T(x+t) - 2\sigma^T(x) + \sigma^T(x-t)) \, dt \right] \right| dx$$

$$\leqslant \int_\alpha^\beta \omega(\sigma^T; x; \delta) \, dx + \mu \int_\delta^\pi \left(\frac{\sin(mt/2)}{m \sin(t/2)} \right)^{2r}$$

$$\times \left\{ \int_\alpha^\beta [\sigma^T(x+t) - 2\sigma^T(x) + \sigma^T(x-t)] \, dx \right\} dt$$

$$\leqslant \int_\alpha^\beta \omega(\sigma^T; x; \delta) \, dx + 2\mu \int_\delta^\pi \left(\frac{\sin(mt/2)}{m \sin(t/2)} \right)^{2r} \omega_L(\sigma^T; t) \, dt, \tag{7}$$

where $\omega(f; x; \delta)$ is the so-called local modulus of continuity of f,

$$\omega(f; x; \delta) = \sup_{|h| \leqslant \delta} | f(x+h) - f(x)|. \tag{8}$$

The definition (8) can be found in [15].

$\omega_L(f; \delta)$ is the integral modulus of continuity of f in the integral $[-\pi, \pi]$,

$$\omega_L(f; \delta) = \sup_{0 < h \leqslant \delta} \int_{-\pi}^\pi | f(x+h) - f(x)| \, dx.$$

Let us estimate the second term in the last line of (7). It is easily seen that for each integrable function f, if $t/u \geqslant 1$, then

$$\omega_L(f; t) \leqslant (2t/u) \, \omega_L(f; u). \tag{9}$$

Also

$$\sin(t/2) \geqslant t/\pi \qquad \text{for} \quad 0 \leqslant t \leqslant \pi. \tag{10}$$

Using (9), (10) and (5), and setting $u = \delta$,

$$\frac{\pi^2 e}{2m} \leqslant \delta \leqslant \frac{\pi^2 e}{2(m-1)},$$

we obtain

$$2\mu \int_{\delta}^{\pi} \left(\frac{\sin(mt/2)}{m \sin(t/2)}\right)^{2r} \omega_L(\sigma^T; t)\, dt \leqslant \frac{m}{\pi}\, \omega_L(\sigma^T; \delta) \left(\frac{\pi^2}{2m}\right)^{2r} \frac{\delta^{-2r+1}}{2r-2}$$

$$= \frac{\pi \cdot \omega_L(\sigma^T; \delta)}{2(2r-2)} \left(\frac{\pi^2}{2m\delta}\right)^{2r-1}$$

$$\leqslant \frac{\pi^3 e^{-2r+2}}{(2r-2)}\, \omega_L(\sigma^T; m^{-1}). \tag{11}$$

From (7) and (11) we obtain

$$\int_{\alpha}^{\beta} | U_{m,r}(\sigma^T; x) - \sigma^T(x)|\, dx$$

$$\leqslant \int_{\alpha}^{\beta} \omega(\sigma^T; x; \delta)\, dx + 2\, \frac{\pi^3 e^{-2r+2}}{(2r-2)} \cdot \frac{1}{m}, \tag{12}$$

since $\omega_L(\sigma^T; m^{-1}) = 2/m$.

Put now $\alpha = 0$, $\beta = (\pi/2) - \delta$. Then from (12):

$$\int_{0}^{(\pi/2)-\delta} | U_{m,r}(\sigma^T; x) - \sigma^T(x)|\, dx \leqslant \frac{\pi^3 e^{-2r+2}}{r-1} \cdot m^{-1}, \tag{13}$$

since $\omega(\sigma^T; x; \delta) = 0$ for $0 \leqslant x < (\pi/2) - \delta$.

Further, put in the integral (13) $x = \arccos y$. Then we obtain:

$$\int_{\delta}^{1} | \sigma(y) - P(y)|\, dy \leqslant \int_{\sin\delta}^{1} | \sigma(y) - P(y)|\, dy$$

$$\leqslant \int_{\sin\delta}^{1} \frac{| \sigma^T(\arccos y) - U_{m,r}(\arccos y)}{(1-y^2)^{1/2}}\, dy$$

$$\leqslant \int_{0}^{(\pi/2)-\delta} | U_{m,r}(\sigma^T; x) - \sigma^T(x)|\, dx$$

$$\leqslant \frac{\pi^3 e^{-2r+2}}{r-1} \cdot m^{-1},$$

where

$$\begin{aligned} \sigma(y) &= 0, \qquad -1 \leqslant y \leqslant 0, \\ &= 1, \qquad 0 < y \leqslant 1, \end{aligned} \tag{14}$$

and $P(y) = U_{m,r}(\arccos y)$, an even algebraic polynomial of degree $\leqslant (m-1) \cdot r$.

P is monotone increasing in $[-1, 1]$ since $U_{m,r}(\sigma; x)$ is a bell-shaped trigonometric polynomial, by means of the transformation $x = \arccos y$ it is mapped into a monotone algebraic polynomial.

On the other hand, we can assume $0 \leqslant P(x) \leqslant 1$ for $x \in [-1, 1]$ since the operator $U_{m,r}(f; x)$ is positive and

$$0 \leqslant \sigma^T(x) \leqslant 1 \qquad \text{for} \quad x \in [-\pi, \pi],$$
$$0 \leqslant \sigma(x) \leqslant 1 \qquad \text{for} \quad x \in [-1, 1].$$

The above considerations and calculations yield

LEMMA 2. *Let σ be the function* (14). *For any positive integers m and r there exists a polynomial P, monotone in* $[-1, 1]$ *and of degree* $\leqslant (m-1)r$, *for which*

$$\int_\delta^1 |\sigma(x) - P(x)|\, dx \leqslant de^{-r}m^{-1}, \tag{15}$$

$$\int_{-1}^{-\delta} |\sigma(x) - P(x)|\, dx \leqslant de^{-r}m^{-1}, \tag{16}$$

$$0 \leqslant P(x) \leqslant 1 \qquad \text{for} \quad x \in [-1, 1], \tag{17}$$

$$P(-x) = 1 - P(x) \qquad \text{for} \quad x \in [0, 1], \tag{18}$$

where d is an absolute constant and $\pi^2 e/2m \leqslant \delta \leqslant \pi^2 e/2(m-1)$.

(18) is obtained easily; we omit its proof.

Proof of Theorem 1. Form the polynomial

$$Q(x) = \sum_{i=1}^{n} a_i P\left(x - \frac{x_i + x_{i-1}}{2}\right),$$

where P is the monotone polynomial from Lemma 2 of degree $\leqslant (m-1)r$. (For the time being m and r are arbitrary positive integers.)

Choose the coefficients a_i in such a way that

$$Q(x_j) = y_j\,, \qquad j = 1,\dots, n. \tag{19}$$

Write the system (19) in matrix form, as follows: namely,

$$\begin{bmatrix} 1-\Delta_{1,1} & \Delta_{1,2} & \Delta_{1,3} & \cdots & \Delta_{1,n} \\ 1-\Delta_{2,1} & 1-\Delta_{2,2} & \Delta_{2,3} & \cdots & \Delta_{2,n} \\ \vdots & \vdots & \vdots & & \vdots \\ 1-\Delta_{n,1} & 1-\Delta_{n,2} & 1-\Delta_{n,3} & \cdots & 1-\Delta_{n,n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \tag{20}$$

where the following notation has been used:

$$1 - \Delta_{1,1} = P\left(x_1 - \frac{x_1 + x_0}{2}\right), \qquad \Delta_{1,2} = P\left(x_1 - \frac{x_2 + x_1}{2}\right), \ldots,$$

$$\Delta_{1,n} = P\left(x_1 - \frac{x_n + x_{n-1}}{2}\right)$$

. . . .

$$1 - \Delta_{n,1} = P\left(x_n - \frac{x_1 + x_0}{2}\right), \quad 1 - \Delta_{n,2} = P\left(x_n - \frac{x_2 + x_1}{2}\right), \ldots,$$

$$1 - \Delta_{n,n} = P\left(x_n - \frac{x_n + x_{n-1}}{2}\right). \tag{21}$$

From (20), after a simple manipulation we get

$$\begin{bmatrix} 1-\Delta_{1,1} & \Delta_{1,2} & \Delta_{1,3} & \cdots & \Delta_{1,n} \\ -\Delta_{2,1}+\Delta_{1,1} & 1-\Delta_{2,2}-\Delta_{1,3} & \Delta_{2,3}-\Delta_{1,3} & \cdots & \Delta_{2,n}-\Delta_{1,n} \\ -\Delta_{3,1}+\Delta_{2,1} & -\Delta_{3,2}+\Delta_{2,2} & 1-\Delta_{3,3}-\Delta_{2,3} & \cdots & \Delta_{3,n}-\Delta_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\Delta_{n,1}+\Delta_{n-1,1} & -\Delta_{n,2}+\Delta_{n-1,2} & -\Delta_{n,3}+\Delta_{n-1,3} & \cdots & 1-\Delta_{n,n}-\Delta_{n-1,n} \end{bmatrix}$$

$$\times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \\ \vdots \\ \Delta y_n \end{bmatrix}. \tag{22}$$

From Lemma 1 it follows that if

$$2 \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |\Delta_{i,j}| \frac{A + B}{B} < 1, \tag{23}$$

then the system (22) has a positive solution.

From (21), (18), (17), and the condition for monotony of the polynomial P we find

$$
\begin{aligned}
&\Delta_{i,1} + \Delta_{i,2} + \cdots + \Delta_{i,n}\\
&\quad = 1 - P\left(x_i - \frac{x_1 + x_0}{2}\right) + \cdots + 1 - P\left(x_i - \frac{x_i + x_{i-1}}{2}\right)\\
&\qquad + P\left(x_i - \frac{x_{i+1} + x_i}{2}\right) + \cdots + P\left(x_i - \frac{x_n + x_{n-1}}{2}\right)\\
&\quad = P\left(\frac{x_1 + x_0 - x_i}{2}\right) + \cdots + P\left(\frac{x_i + x_{i-1}}{2} - x_i\right)\\
&\qquad + P\left(x_i - \frac{x_{i+1} + x_i}{2}\right) + \cdots + P\left(x_i - \frac{x_n + x_{n-1}}{2}\right)\\
&\quad \leqslant P(x_1 - x_i) + \cdots + P(x_{i-1} - x_i) + P\left(\frac{x_i + x_{i-1}}{2} - x_i\right)\\
&\qquad + P\left(x_i - \frac{x_{i+1} + x_i}{2}\right) + P(x_i - x_{i+1}) + \cdots + P(x_i - x_{n-1}).
\end{aligned} \tag{24}
$$

Let $x_1 - x_i = \xi_1$, $x_2 - x_i = \xi_2, \ldots, x_{i-1} - x_i = \xi_{i-1}$, $(x_i + x_{i-1})/2 - x_i = \xi_i$, $[(x_i + x_{i-1})/2 - x_i]/2 = \xi_{i+1}$.

Then since $\min\{(\xi_2 - \xi_1), \ldots, (\xi_i - \xi_{i-1}), (\xi_{i+1} - \xi_i)\} \geqslant \min_{1 \leqslant j \leqslant n}(x_j - x_{j-1})/4 \geqslant c_0 n^{-1}/4$,

$$
\begin{aligned}
&P(x_1 - x_i) + P(x_2 - x_i) + \cdots + P(x_{i-1} - x_i) + P\left(\frac{x_i + x_{i-1}}{2} - x_i\right)\\
&\quad = P(\xi_1)\frac{\xi_2 - \xi_1}{\xi_2 - \xi_1} + \cdots + P(\xi_{i-1})\frac{\xi_i - \xi_{i-1}}{\xi_i - \xi_{i-1}} + P(\xi_i)\frac{\xi_{i+1} - \xi_i}{\xi_{i+1} - \xi_i}\\
&\quad \leqslant \frac{4}{\min_{1 \leqslant j \leqslant n}(x_j - x_{j-1})}\,[P(\xi_1)(\xi_2 - \xi_1) + \cdots + P(\xi_{i-1})(\xi_i - \xi_{i-1})\\
&\qquad + P(\xi_i)(\xi_{i+1} - \xi_i)]\\
&\quad \leqslant \frac{4n}{c_0}\int_{-1}^{-c_0/4n} [P(x) - \sigma(x)]\,dx
\end{aligned} \tag{25}
$$

and in the same way

$$
\begin{aligned}
&P(x_i - x_{n-1}) + \cdots + P(x_i - x_{i+1}) + P\left(x_i - \frac{x_{i+1} + x_i}{2}\right)\\
&\qquad \leqslant \frac{4n}{c_0}\int_{-1}^{-c_0/4n} [P(x) - \sigma(x)]\,dx.
\end{aligned}
$$

From (24), (25), and (26) we find

$$\max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |\Delta_{i,j}| \leqslant \frac{8n}{c_0} \int_{1}^{-c_0/4n} [P(x) - \sigma(x)]\, dx.$$

Now we set $m = [2\pi^2 en/c_0] + 1$, $\delta = c_0/4n$. Then $\pi^2 e/2m \leqslant \delta = c_0/4n \leqslant \pi^2 e/2(m-1)$ and Lemma 2 gives:

$$2 \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |\Delta_{i,j}| \leqslant \frac{16n}{c_0} de^{-r}m^{-1} \leqslant d_1 e^{-r},$$

where d_1 is a constant depending only on c_0.

There exists a constant d_2 depending only on c_0 (in a way which is clear from the above inequality and (23)), so that if $r = [d_2 \ln(e + A/B)]$, then

$$d_1 e^{-r} \cdot \frac{A + B}{B} < 1;$$

or, if $m = [2\pi^2 en/c_0] + 1$ and $r = [d_2(c_0) \ln(e + A/B)]$, then (23) holds and the system (20), hence the system (19), has a positive solution.

The interpolation polynomial $Q(x) = \sum_{i=1}^{n} a_i P(x - (x_i + x_{i-1})/2)$ will be monotone since P is monotone in $[-1, 1]$, and $a_i \geqslant 0$, $i = 1,..., n$. The degree of Q equals that P, i.e., it equals $(m-1)$. $r \leqslant c[n \ln(e + A/B)]$, where c is a constant depending only on the constant c_0.

Thus Theorem 1 is proved.

The estimate for the degree of the monotone interpolation polynomial: $cn \ln(e + A/B)$ of Theorem 1 yields the results of [4, 6]: For $A/B \leqslant$ const., the result in [4] is obtained and for $A/B \leqslant n^\alpha$, $\alpha \geqslant 1$, the result in [6].

Of interest are the cases where A/B lies between a constant and a constant times n^α, $\alpha \geqslant 1$. In all these cases Theorem 1 estimates the degree of the monotone interpolation polynomial. For example, if $A/B \leqslant \ln n$, the degree of the monotone interpolation polynomial is $\leqslant cn \ln \ln n$.

The exactness of the estimate of Theorem 1 follows from [4, 6] only if $A/B \leqslant$ const. or $A/B \leqslant n^\alpha$, $\alpha \geqslant 1$. However, it proved that Theorem 1 gives an exact answer to the question of the degree of the monotone interpolation polynomial for an arbitrary order of A/B. From Theorem 1 one can also easily obtain the result of Lorentz and Zeller [8].

From Theorem 2 one can prove the proposition in [13] but we do not consider this here since Theorem 2 is not proved in this paper.

From Theorem 1 it is also possible to obtain an estimate for the polynomial monotone approximation of a bounded monotone function with respect to the Hausdorff distance.

Hausdorff distance between functions was introduced by Sendov and Penkov [17]. Let f and g be functions, bounded in $[0, 1]$.

The number

$$r(f, g) = \max\{\sup_{A\in\bar{f}} \inf_{B\in\bar{g}} d(A, B), \sup_{A\in\bar{g}} \inf_{B\in\bar{f}} d(A, B)\},$$

where $d(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$, $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $\bar{f}$ is the completed graph of the function f (the intersection of all closed point sets, bounded along the y-axis which are convex with respect to the y-axis, whose projection on the x-axis coincides with $[0, 1]$ and which contain the graph of f) is called the Hausdorff distance between f and g.

The number

$$E_n(f)_r = \inf_{P\in H_n} r(f, P)$$

is called the best approximation of f with respect to the Hausdorff distance.

A basic result in the theory of Hausdorff approximation is the following universal estimate obtained by Sendov [18]:

$$E_n(f)_r = O(\ln n/n).$$

From Theorem 1 it is easily obtained that the order of the above estimate is preserved when we approximate a monotone bounded function by a monotone polynomial with respect to the Hausdorff distance. This fact was proved for the first time in [16].

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